ENTRANCE TEST FOR Ph.D. PROGRAMME, 2023

MATHEMATICS

Time : Three Hours Maximum : 100 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

Choose the correct answer from the choices given :

1. Let $A = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$ and $B = \{(x, y) \in \mathbb{R}^2; xy = 1\}$. Then :

   (A) $A \cup B = \mathbb{R}$.   (B) $A \cap B = \phi$.

   (C) $A \cap B \neq \phi$.   (D) $A \cup B = A$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. The $f([-3, -1) \cup [0, 2])$ is :

   (A) $[0, 9]$.   (B) $[0, 16]$.

   (C) $[0, 4]$.   (D) $\{1\}$.

3. Which of the following is a countable set ?

   (A) The collection of all finite subsets of $\mathbb{N}$.

   (B) The set $2^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to $\{0, 1\}$.

   (C) The set of transcendental numbers.

   (D) The set of irrational numbers.

4. The value of $e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$ at the point $x = \pi/4$ is :

   (A) $1/2$.   (B) $\frac{1}{\sqrt{2}}$.

   (C) $\sqrt{2}$.   (D) 2.

5. If $A = \begin{bmatrix} 8 & 5 \\ 7 & 6 \end{bmatrix}$, then the value of $|A^{121} - A^{120}|$ is :

   (A) 120.   (B) 1.

   (C) 121.   (D) 0.

6. The eigen values of the matrix $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ are :

(A) $1, 2$.
(B) $-2, 1$.
(C) $5, -2$.
(D) $3, -2$.

7. The minimal polynomial of the matrix $A = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ is :

(A) $t^2 - 3t + 2$.
(B) $(t-2)(t-1)^2$.
(C) $t^2 + 3t - 2$.
(D) $(t-2)^3$.

8. Let $\mathbb{Z}_n^*$ be the set of integers from 1 to $n - 1$ that are relatively prime to n. The order of the group $G = < \mathbb{Z}_{21}^*, \times >$ is :

(A) 4.
(B) 12.
(C) 6.
(D) 10.

9. The set $B(x_0; r) = \{x \in X : d(x, x_0) \leq r\}$ with centre x_0 and radius r is called :

(A) Open ball.
(B) Sphere.
(C) Open circle.
(D) Closed ball.

10. Which of the following is not a property of norm in general ?

(A) $\|x\| \geq 0$.
(B) $\|x + y\| \leq \|x\| + \|y\|$.
(C) $\|kx\| \leq k\|x\|$.
(D) $\|x\| = 0$ iff $x = 0$.

11. Which of the following subsets of $\mathbb{R}$ is not closed with respect to the usual metric for $\mathbb{R}$?

(A) $\mathbb{N}$.
(B) $\{1, 2, 3\}$.
(C) $(2,3] - \{2\frac{1}{2}\}$.
(D) $[1, 3]$.

12. For $X = \{a, b, c\}$, which of the following is not a topology on X :

(A) $T = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, X\}$.

(B) $T = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, X\}$.

(C) $T = \{\phi, \{b\}, \{a,b\}, X\}$.

(D) $T = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$.

13. The set of points $z \in \mathbb{C}$ for which $|z-2|+|z+2i|=4$ is the conic :

(A) Hyperbola. (B) Rectangle.

(C) Square. (D) Ellipse.

14. The function $w = e^z$ is :

(A) Entire function. (B) Entire non-periodic.

(C) Neither entire nor periodic. (D) Periodic but not entire.

15. The value of the integral $\int_C \frac{z^2-z+1}{z-1} dz$, where C is the circle $|z| = \frac{1}{2}$, is :

(A) $2\pi i$. (B) $-2\pi i$.

(C) 1. (D) 0.

16. The radius of convergence of the series $\sum_{n=0}^{\infty} \frac{z^n}{n}$ is :

(A) 0. (B) 1.

(C) ∞. (D) None of these.

17. For the integral $\int_0^{\infty} \frac{\sin mt}{t} dt = \alpha, m > 0$:

(A) $\alpha = 0$. (B) $\alpha = \frac{\pi}{2}$.

(C) $\alpha = \pi$. (D) $-\pi$.

18. The improper integral $\int_0^{\infty} x^{m-1} e^{-x} dx$ is convergent :

(A) Only if $m > 0$. (B) If $m > 0$.

(C) If and only if $m > 0$. (D) For all values of m.

19. If $\mathcal{L}[f(t)] = F(p)$, then $\mathcal{L}[f(\alpha t)] = kF\left(\frac{p}{\alpha}\right), \alpha \neq 0$, where k is :

(A) α. (B) $\frac{1}{\alpha}$.

(C) α^2. (D) $\frac{1}{\alpha^2}$.

Turn over

20. The last unit digit of 3^{400} is :

(A) 2. (B) 0.

(C) 1. (D) 3.

21. The quadratic residue of Modulo 23 is :

(A) 1, 2, 3, 5, 6, 7, 12. (B) 1, 2, 3, 4, 8, 9, 12, 13, 16, 18.

(C) 1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18. (D) 1, 2, 5, 6, 7, 8, 9, 13, 16, 18.

22. The matrix $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}$ corresponds to the quadratic form :

(A) $2x^2 + 3y^2 + 4z^2 + 2xy - 10xz + 4yz$.

(B) $2x^2 + 3y^2 + 4z^2 + 2xy - 10xz - 4yz$.

(C) $2x^2 + 3y^2 + 4z^2 - 2xy + 10xz - 4yz$.

(D) $2x^2 + 3y^2 + 4z^2 + 2xy + 10xz - 4yz$.

23. For what values of λ, μ the equations $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ have no solution :

(A) $\lambda = 3, \mu = 10$. (B) $\lambda = 3, \mu \neq 10$.

(C) $\lambda \neq 3, \mu = 10$. (D) $\lambda \neq 3$.

24. Let N be a subgroup of a group G. Then :

(A) $N \trianglelefteq G$.

(B) $xNx^{-1} = N$ for every $x \in G$.

(C) $(xN)(yN) = xyN$ for all $x, y \in G$.

(D) All options (a), (b), (c) are correct.

25. Let G be a group of order 6 and H be a subgroup of G such that $1 < |H| < 6$. The correct option is :

(A) G is always cyclic, but H may not be cyclic.

(B) G may not be cyclic but H is always cyclic.

(C) Both G and H are always cyclic.

(D) Both G and H may not cyclic.

26. Let G be a group order 15. The number of Sylow subgroup of G of order 3 is :

(A) 1. (B) 3.

(C) 5. (D) 7.

27. $\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n}$ is :

(A) 1. (B) $\frac{4}{e}$.

(C) $\frac{1}{e}$. (D) e^2.

28. Let f and g be functions defined on [0, 1] as follows :

$$f(x) = 1, \quad \text{when } x \text{ is rational}$$
$$= 0, \quad \text{when } x \text{ is irrational}$$

and

$$g(x) = \sin\left(\frac{1}{x}\right), \quad \text{when } x \text{ is irrational}$$
$$= 0, \quad \text{otherwise.}$$

Then :

(A) f is Riemann integrable but g is not.

(B) g is Riemann integrable but f is not.

(C) Both f and g are Riemann integrable.

(D) Both f and g are not Riemann integrable.

29. Which of the following is not a vector space :

(A) $\mathbb{C}$ over $\mathbb{R}$. (B) $\mathbb{R}$ over $\mathbb{C}$.

(C) $\mathbb{C}$ over $\mathbb{Q}$. (D) $\mathbb{Q}$ over $\mathbb{R}$.

30. If n objects are put in m boxes and $n > m$, then :

(A) At least one box will contain exactly two objects.

(B) At least one box will contain less than two objects.

(C) At least one box will contain exactly two or more objects.

(D) At least one box will contain less than and equal to two objects.

31. Which of the following is not a field ?

(A) $\frac{\mathbb{Q}[x]}{<x^2-5x+6>}$. (B) $\frac{F[x]}{<x^2+x+2>}$.

(C) $\frac{\mathbb{Z}_3[x]}{<x^3+2x+1>}$. (D) $\frac{\mathbb{Z}_2[x]}{<x^2+x+1>}$.

32. The units of $\mathbb{Z}[\sqrt{-5}]$ is/are :

(A) 1.
(B) -1.
(C) $0, \pm 1$.
(D) ± 1.

33. If $u = f(x-y, y-z, z-x)$, then the value of $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$ is :

(A) 3.
(B) 0.
(C) $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}$.
(D) None of these.

34. The Laplace's equation in cylindrical co-ordinates is :

(A) $\frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial^2}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}+\frac{\partial^2}{\partial z^2}=0.$

(B) $\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{k}\frac{\partial u}{\partial t}.$

(C) $\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$

(D) $\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2}+\frac{\partial^2 u}{\partial z^2}=0.$

35. Let T be a linear operator on $\mathbb{R}^2$ defined by T $(x, y) = (4x - 2y, 2x + y)$. The matrix of T relative to the basis $\{(1, 1), (-1, 0)\}$ is :

(A) $\begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix}$.
(B) $\begin{bmatrix} 3 & -2 \\ 1 & -2 \end{bmatrix}$.
(C) $\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$.
(D) $\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$.

36. The general solution of the differential equation $(2xy + e^y)\,dx + (x^2 + xe^x)\,dy = 0$ is :

(A) $xy + xe^y = c$, where c is constant.
(B) $x^2y - xe^y = c$, where c is constant.
(C) $x^2y + xe^y = c$, where c is constant.
(D) $ye^x + xe^y = c$, where c is constant.

37. The partial differential equation

$$u_{xx} - 2(\sin x)u_{xy} - \left(\cos^2 x\right)u_{yy} - (\cos x)u_y = 0.$$

(A) Hyperbolic. (B) Parabolic.

(C) Elliptic. (D) None of these.

38. In a random experiment $P(A) = \frac{1}{12}$, $P(B) = \frac{5}{12}$ and $P(B \mid A) = \frac{1}{15}$, then P (A ∪ B) is :

(A) $\frac{4}{52}$. (B) $\frac{1}{180}$.

(C) $\frac{3}{28}$. (D) $\frac{89}{180}$.

39. The value of the integral $\int_S \vec{r}.d\vec{S}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and S is the surface of a sphere of Radius R, is :

(A) $4\pi R^2$. (B) πR^3.

(C) $\frac{3}{4}\pi R^3$. (D) $\frac{3}{4}\pi R^2$.

40. $\lim_{x \to 0} \frac{\sin x - x}{x^3}$ is :

(A) $\frac{1}{6}$. (B) $-\frac{1}{6}$.

(C) $\frac{1}{2}$. (D) $-\frac{1}{2}$.

41. The lim *sup* and lim *inf* of the sequence $\left\{(-1)^n \left(1 + \frac{1}{n}\right)\right\}$ are :

(A) 2, 1. (B) 1, 1.

(C) 1, – 1. (D) 5, 1.

42. If $\sum u_n$ is a series of positive terms, then :

(A) Convergence of $\sum (-1)^n u_n \Rightarrow$ convergence of $\sum u_n$.

(B) Convergence of $\sum u_n \Rightarrow$ convergence of $\sum (-1)^n u_n$.

(C) Convergence of $\sum(-1)^n u_n \Rightarrow$ divergence of $\sum u_n$.

(D) Divergence of $\sum u_n \Rightarrow$ divergence of $\sum(-1)^n u_n$.

43. The area of triangle whose two sides are represented by the vectors $3\hat{i}+4\hat{j}$ and $5\hat{i}+7\hat{j}+\hat{k}$ is :

(A) $\frac{\sqrt{26}}{3}$. (B) $\sqrt{26}$.

(C) 13. (D) $\frac{\sqrt{13}}{2}$.

44. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors (1, – 2, 5, – 3), (2, 3, 1, – 4), (3, 8, – 3, – 5). The dimension of W is :

(A) 1. (B) 2.

(C) 3. (D) 4.

45. Which of the following is not interpolation formula :

(A) Lagrange. (B) Hermite.

(C) Gauss-Seidal. (D) Spline.

46. The Saddle point of the function $f(x,y)=3x^2-y^2+x^3$ is :

(A) $(2,0)$. (B) $(4\cdot 0)$.

(C) $(2,-1)$. (D) None.

47. What is the number of non-singular 3 × 3 matrices over $\mathbb{F}_2$, the finite field with two elements ?

(A) 168. (B) 384.

(C) 2^3. (D) 3^2.

48. The number of elements in a basis $\mathbb{Q}(\sqrt{3},\sqrt{5})$ over $\mathbb{Q}$ is :

(A) 3. (B) 7.

(C) 4. (D) 27.

49. If $f(x)=x^2, -\pi\le x\le\pi$, be represented in Fourier series as $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$, then the value of a_0 is :

(A) $\frac{2\pi^2}{3}$. (B) $\frac{4(-1)^n}{3}$.

(C) 0. (D) 4.

50. The correct statement is :

(A) $S=\{u\in E^n : \|u\|\leq 1\}$ is a convex set.

(B) $S=\{(x_1,x_2,x_3): x_1^2+x_2^2+x_3^2\leq 1\}$ is not a convex set.

(C) $S=\{(x_1,x_2): 2x_1+3x_2=7\}$ is a convex set.

(D) $S=\{(x_1,x_2): 3x_1^2+2x_2^2\leq 6\}$ is a convex set.

($50 \times 1 = 50$ marks)

Part B

Answer any **ten** *questions.*
Each question carries 5 marks.

51. Prove or disprove : Every convergent sequence is bounded but a bounded sequence is not convergent.

52. If $=\cot^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{4}\sin 2u=0$.

53. Show that $x^3 + 2x + 1$ is irreducible over $\mathbb{Z}_3$ and construct a field with 27 elements. Hence or otherwise find the inverse of $x^2 + < x^3 + 2x + 1 >$ in $\mathbb{Z}_3$.

54. Prove that the matrix $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$ satisfies the characteristics equation $A^2 - 4A - 5I = 0$ and hence find A^{-1}.

55. Solve the differential equation $(x+y+z)\frac{dy}{dx}=1$.

56. Test the convergence of $\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n+1}$.

57. Let $f(x) = \frac{1}{n}, \quad \frac{1}{n+1}<x\leq\frac{1}{n}(n=1,2,3,\ldots)$
$= 0, \quad x=0.$

Prove that f is integrable on [0, 1] and hence evaluate $\int_0^1 f$.

58. Solve :

$$x \equiv 3 \pmod 5$$
$$x \equiv 4 \pmod 7$$
$$x \equiv 6 \pmod 9$$

59. A person requires 10, 12 and 12 units of chemicals A, B and C respectively for his garden. A liquid product contians 5, 2 and 1 unit of A, B and C respectively per jar A dry product contains 1, 2 and 4 units of A, B and C respectively per carton. If the liquid product sells for Rs. 30 per jar and the dry product sells for Rs. 35 per carton to minimize the cost and meet the requirements ? Formulate the Linear Programming Problem.

60. Write a Research Proposal in brief based on the study of a recent Mathematical research article.

61. What function do programming languages and software play in modern Mathematical research ?

62. Discuss some metrices and parameters utilized to analyze a journal of Mathematics.

63. Briefly describe the steps involved in carrying out research in Mathematical Science.

64. List a few tools for data analysis and describe the various data analysis methodologies in research.

(10 × 5 = 50 marks)